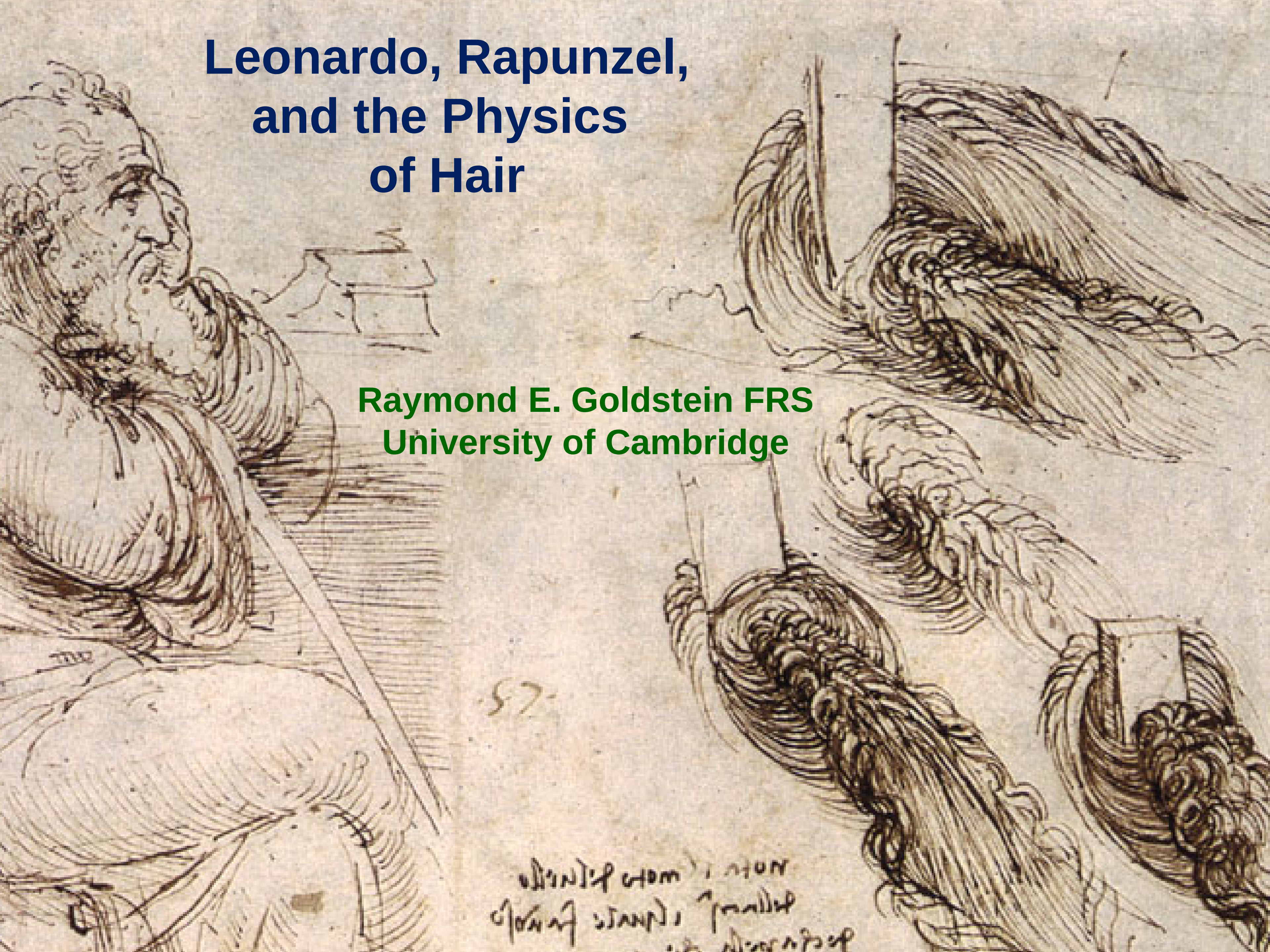
Leonardo, Rapunzel, and the Physics of Hair



Raymond E. Goldstein FRS
University of Cambridge


BRAVE

Super-Helices for Predicting the Dynamics of Natural Hair

Bertails, Audoly, Cani, Querleux, Leroy, Lévêque (SIGGRAPH 2006)

Part 3

Animation
of a full head of hair

Opportunity Knocks…

Dear Prof. Goldstein,

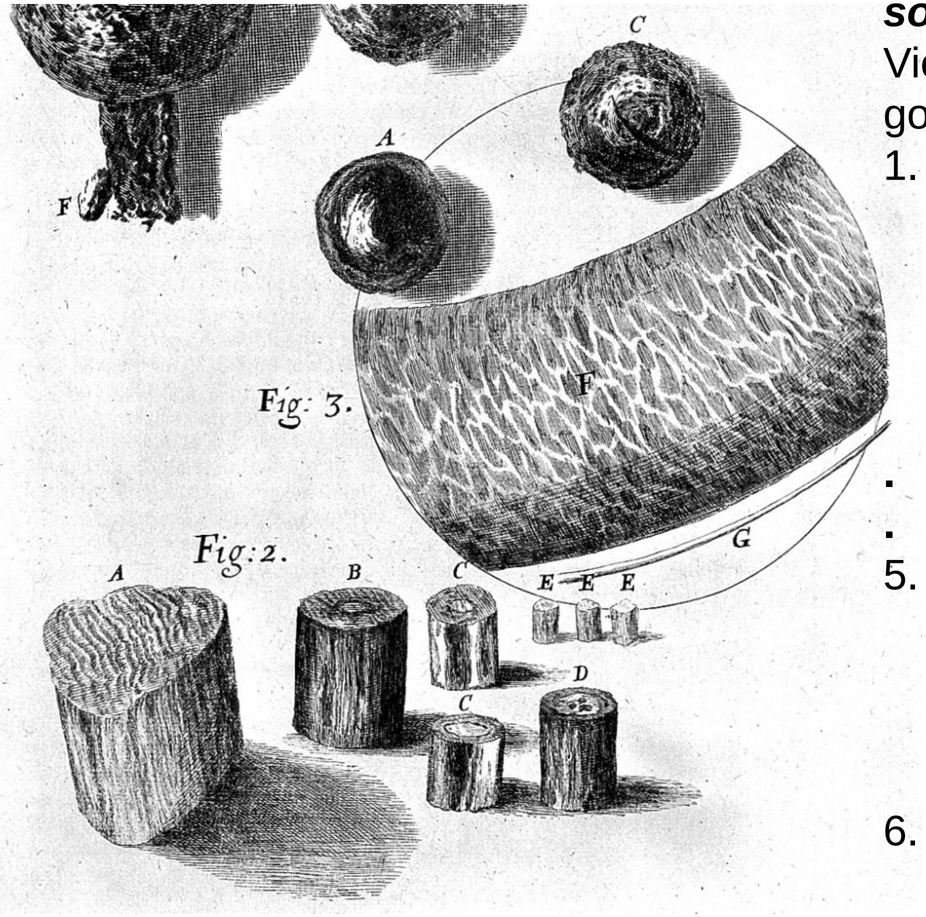
I work at Unilever's R&D labs in Port Sunlight in the UK in the **Hair Research Division**. My personal background being in the Soft Matter Physics area.
Some of the challenging technical problems in the Hair Care area depends upon us better understanding **hair array statistical mechanics** under various conditions. From your publications and your current research interests I see that your research interests lies in quite varied and challenging areas. I was wondering if the area of **hair array statistical mechanics** may be something you might possibly be interested in? …

…

Samiul Amin

Unilever R&D Port Sunlight, Quarry Road East, Bebington, Wirral CH63 3JW

Drawings of Hair, From Hooke’s *Micrographia* (1665)



Wellcome Library, London

Observ. XXXII. ***Of the Figure of several sorts of* Hair*, and of the texture of the* skin*.***

Viewing some of the Hairs of my Head with a very good *Microscope*, I took notice of these particulars:

1. That they were, for the most part, *Cylindrical*, some of them were somewhat *Prismatical*, but generally they were very neer round, such as are represented in the second Figure of the 5. *Scheme*, by the *Cylinders* EEE. nor could I find any that had sharp angules.

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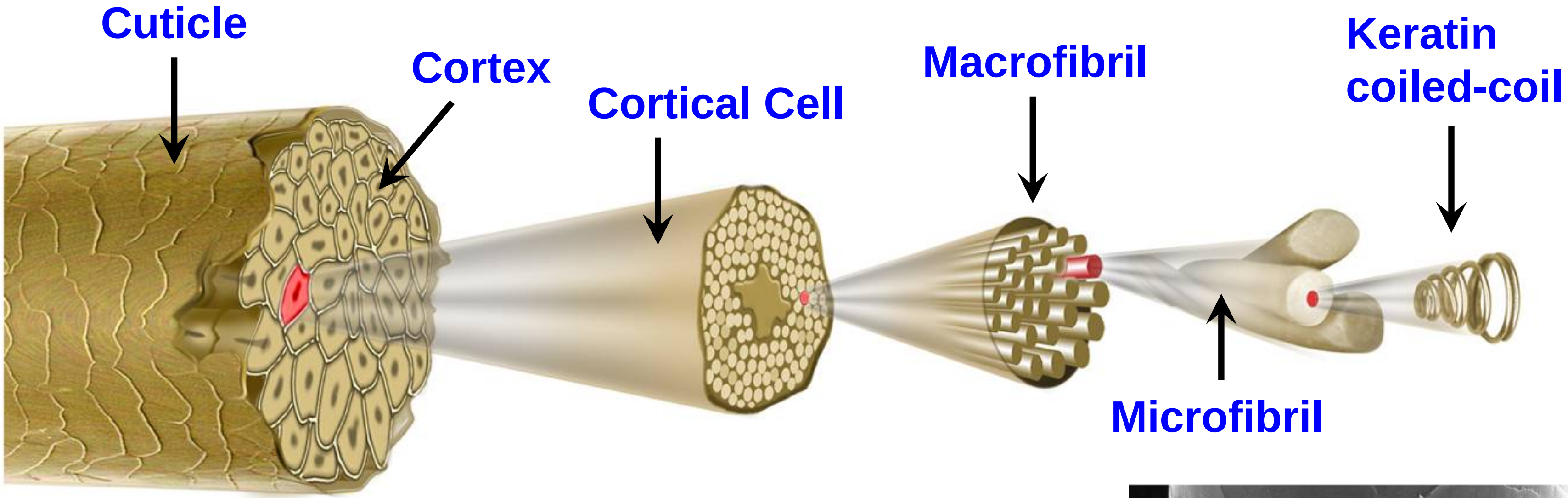
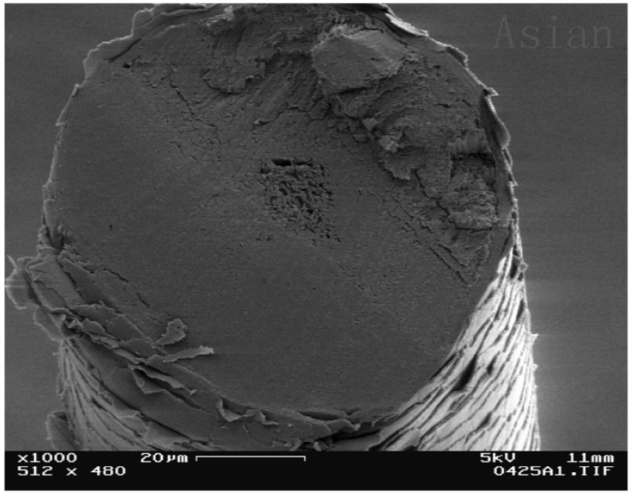
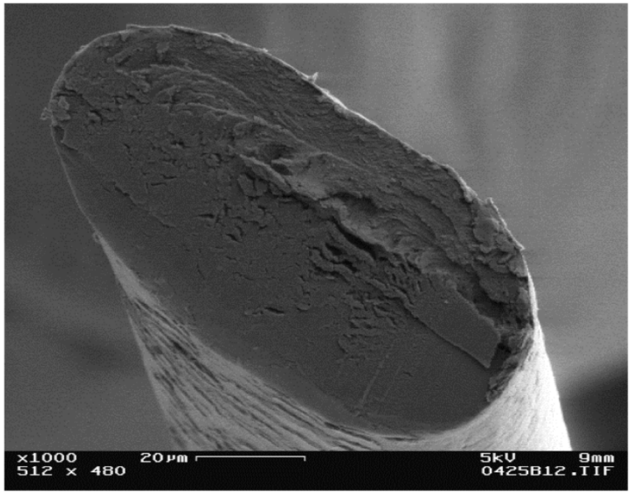
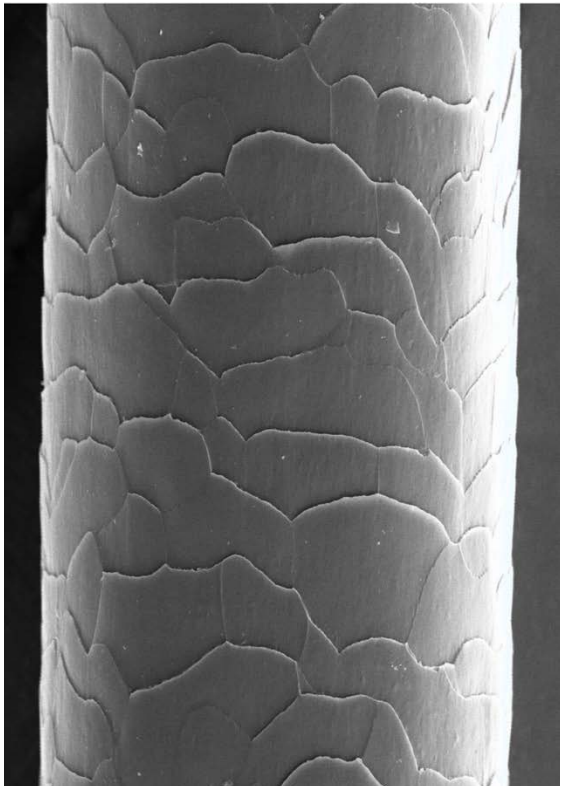
5. That the top when split (which is common in long Hair) appear'd like the end of a stick, beaten till it be all flitter'd, there being not onely two splinters, but sometimes half a score and more.
6. That they were all, as farr as I was able to find, solid *Cylindrical* bodies, not pervious, like a Cane or Bulrush; nor could I find that they had any Pith, or distinction of Rind, or the like, such as I had observ'd in Horse-hairs, the Bristles of a Cat, the *Indian* Deer's Hair, *&c.*

Hair Has a Complex Structure!









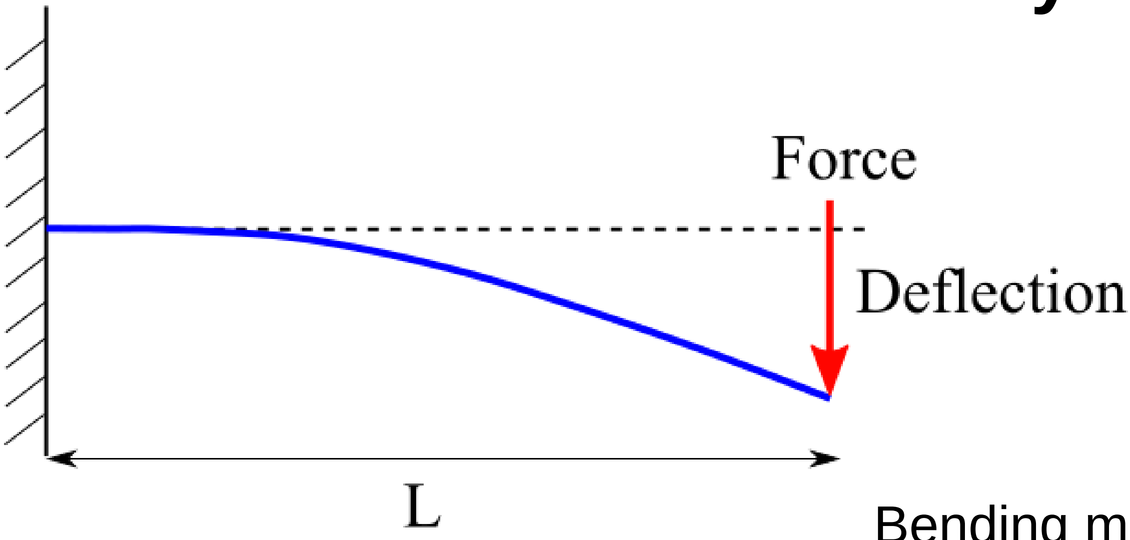
Interesting Facts About Hair

R.E.G.

- Adults have 50,000-100,000 head hairs
- Growth of 1 cm/month ≈ **4 nm/sec** per hair
- Hair has a density of 1.3 g/cm^3, is elliptical in cross-section, with an average major axis diameter **d ≈ 80 μm** and a linear mass density **λ** ≈ 65 μg/cm ≈ **6.5 g/km**

With 100,000 hairs, each of 0.25 m in length, we Have 25,000 m or 25 km of hair on our head!

When Does Gravity Matter?



$$\boxed{D \sim \frac{F L^3}{A}}$$

Bending modulus

If the filament is the right size, say ℓ, that gravity ($F = \lambda \ell g$) can deflect it an amount comparable to its size then

$$\boxed{\ell \sim \frac{F\ell^3}{A} \sim \frac{\lambda g \ell^4}{A}}$$

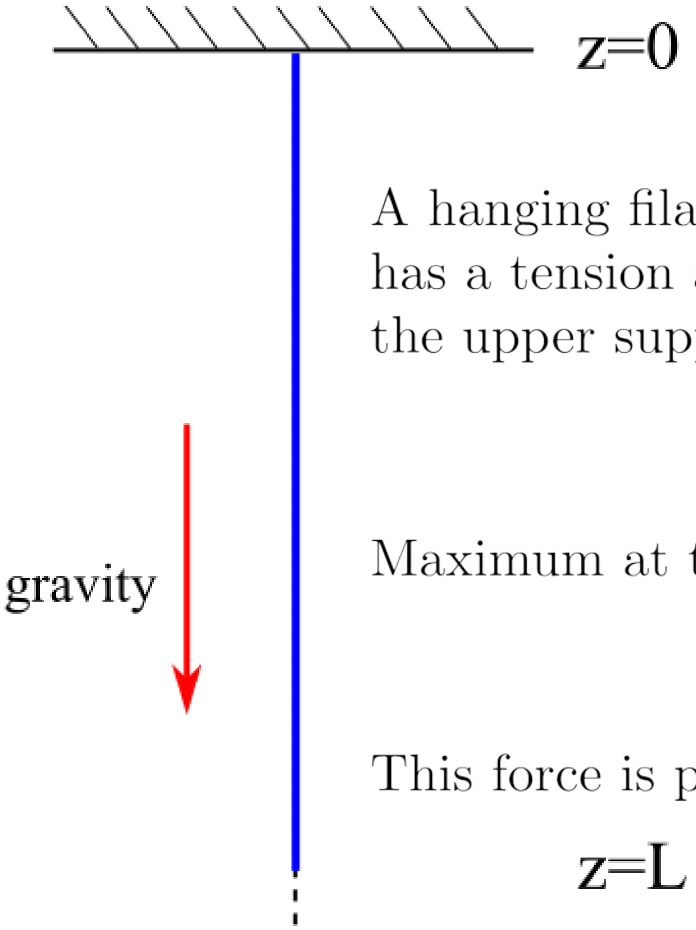
or

$$\ell \sim \left(\frac{A}{\lambda g}\right)^{1/3} \sim 5 \text{ cm}$$

Hence, we introduce the “Rapunzel number”

$$\boxed{Ra \equiv \frac{L}{\ell}}$$

Hanging Hair is Under Tension



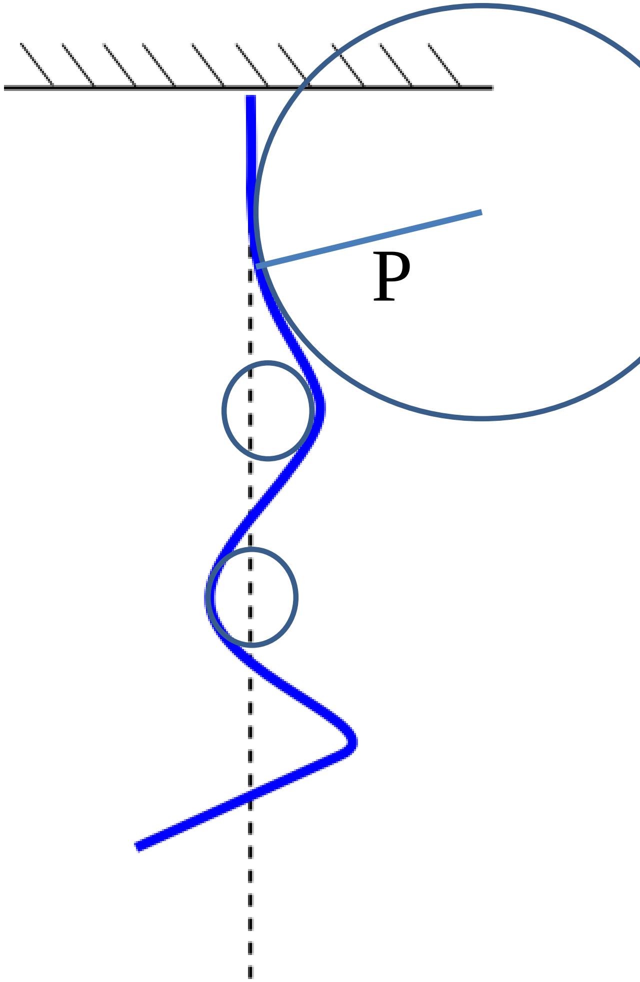
A hanging filament with linear mass density λ has a tension at location z (measured down from the upper support) equal to the weight below:

$$\boxed{T(z) = \lambda g(L - z)}$$

Maximum at the top and vanishes at the lower free end.

This force is parallel to the filament.

Curvature: Euler and Bernoulli



At each point z along the filament we can construct an inscribed circle whose radius $P(z)$ gives the curvature κ as

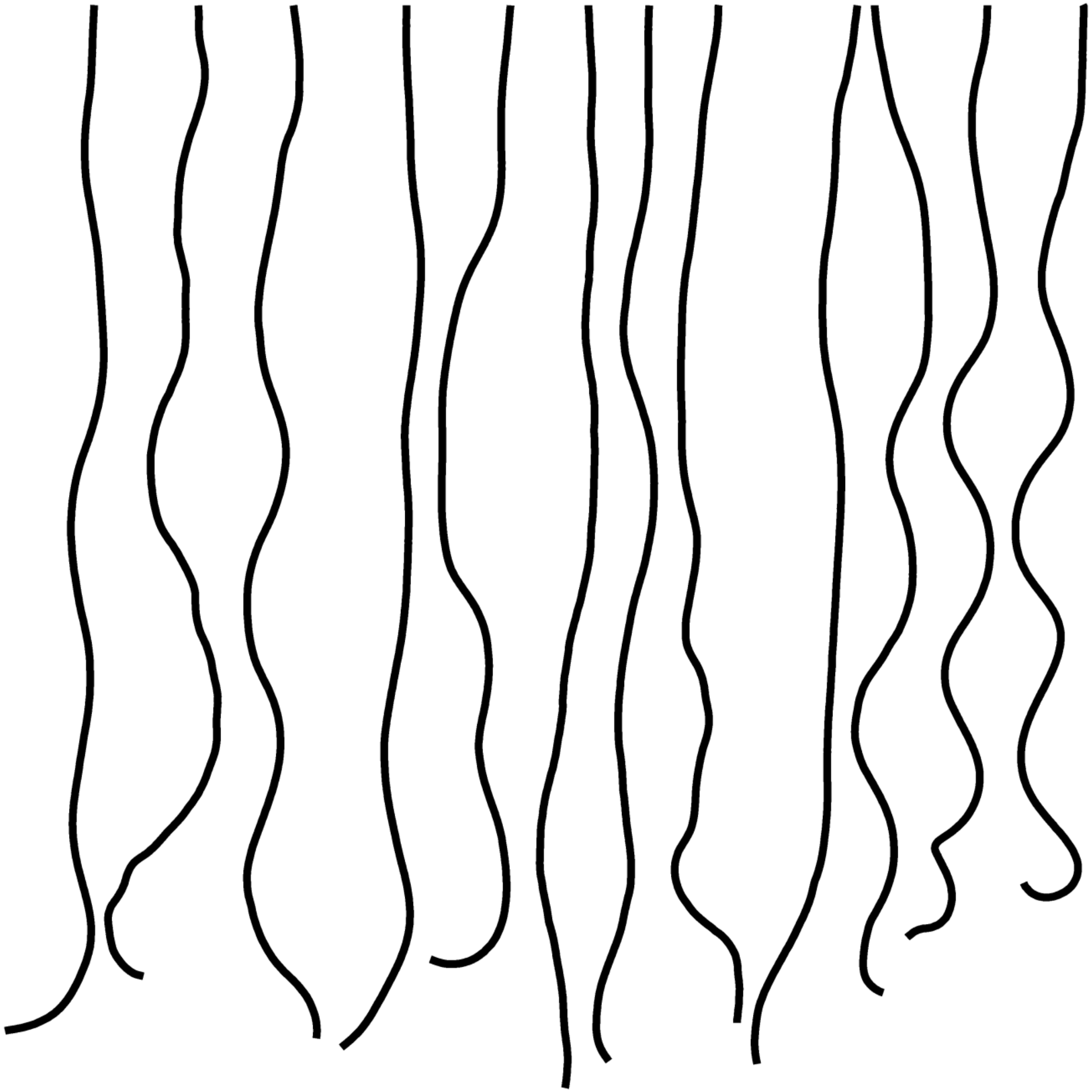
$$\boxed{\kappa = \frac{1}{P}}$$

Euler and Bernoulli showed us that the energy per unit length of an elastic filament is

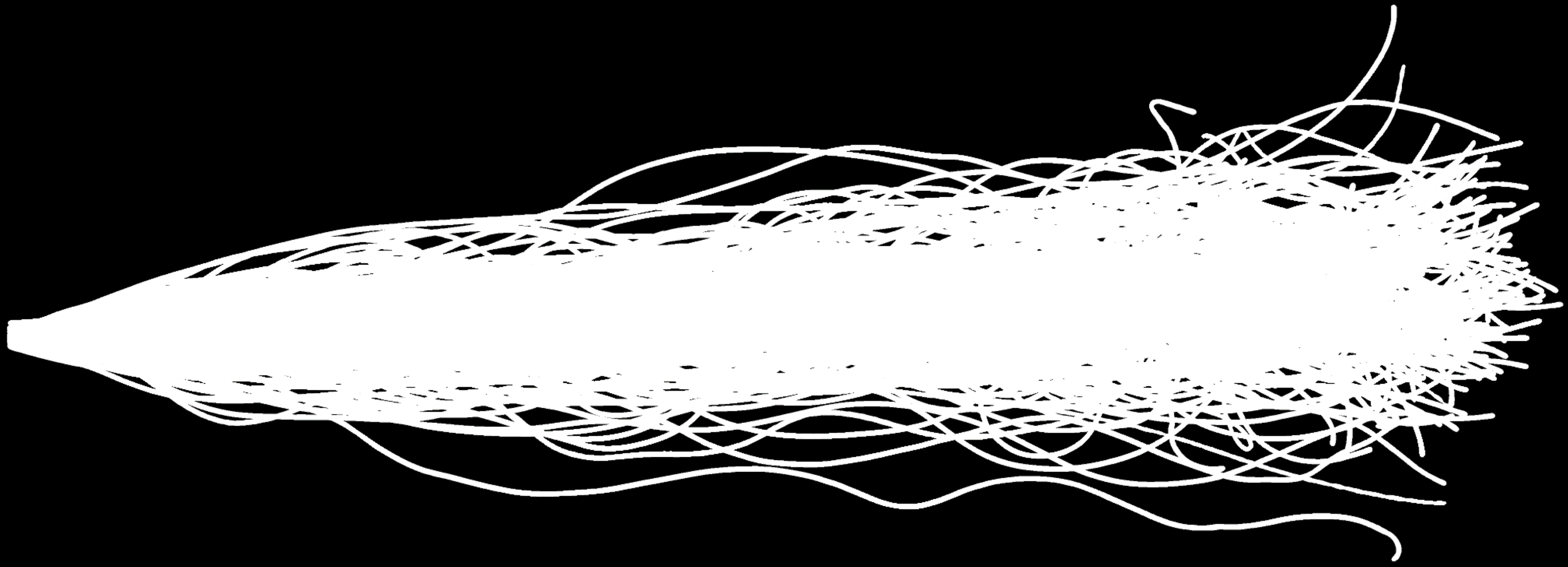
$$\boxed{\frac{1}{2}A\kappa^2}$$

This is clearly minimized for a straight filament.

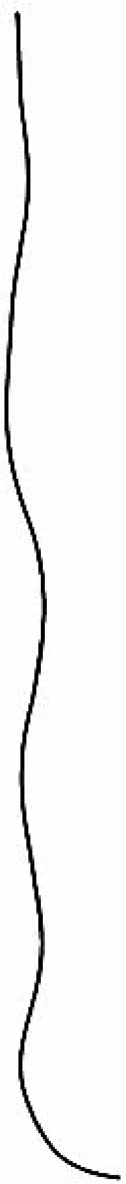
Hair Has Random Intrinsic Curvatures



Overlaid Images of Hairs from a Ponytail

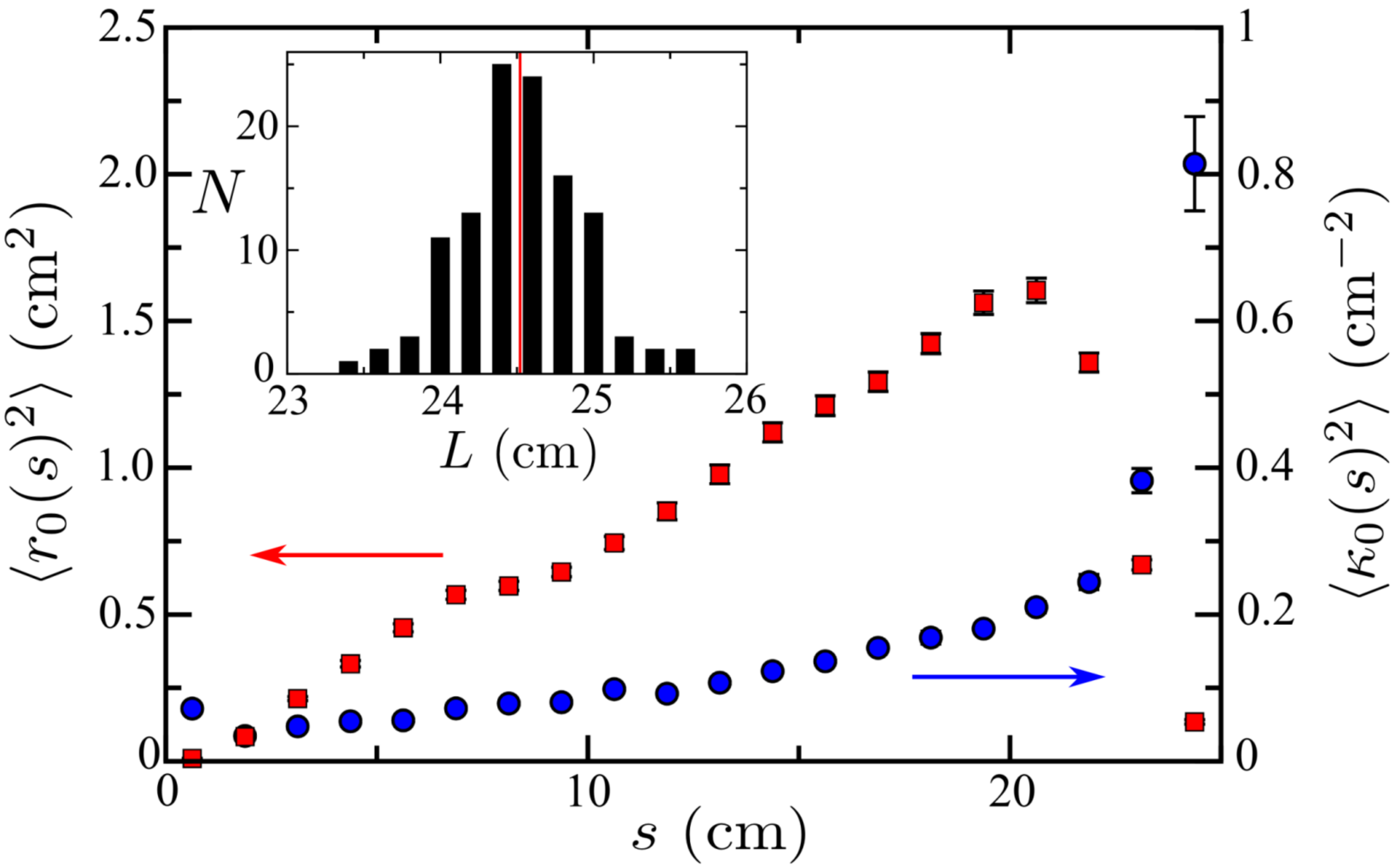


Statistics of Random Curvatures



Measurements on 115 hairs from a commercial* switch, using high-resolution stereographic imaging.

Filament reconstruction based in part on an algorithm due to W.S. Ryu for *C. elegans* tracking.



*International Hair Importers & Products, Inc. (Glendale, NY)

Leonardo’s Observation

“Observe the motion of the surface of the water which resembles that of hair, and has two motions, of which one goes on with the flow of the surface, the other forms the lines of the eddies…”

The Notebooks of Leonardo da Vinci – “On the proportions and on the movements of the human figure”

Density Functional Theory of Fiber Bundles

Fiber length density $\boxed{\rho(\mathbf{r})}$

(#/unit area crossing a plane ⊥ to fibers)

Local mean orientation of hairs $\boxed{\mathbf{t}(\mathbf{r})}$

Absence of free ends → $\boxed{\nabla \cdot (\rho\,\mathbf{t}) = 0}$

Hypothesis: a *local* energy functional,

filament elasticity

disorder

$$\boxed{\mathcal{E} = \int d^3r \rho \left(\tfrac{1}{2} A\kappa^2 + \varphi(\mathbf{r}) + \langle u \rangle \right)}$$

external potential

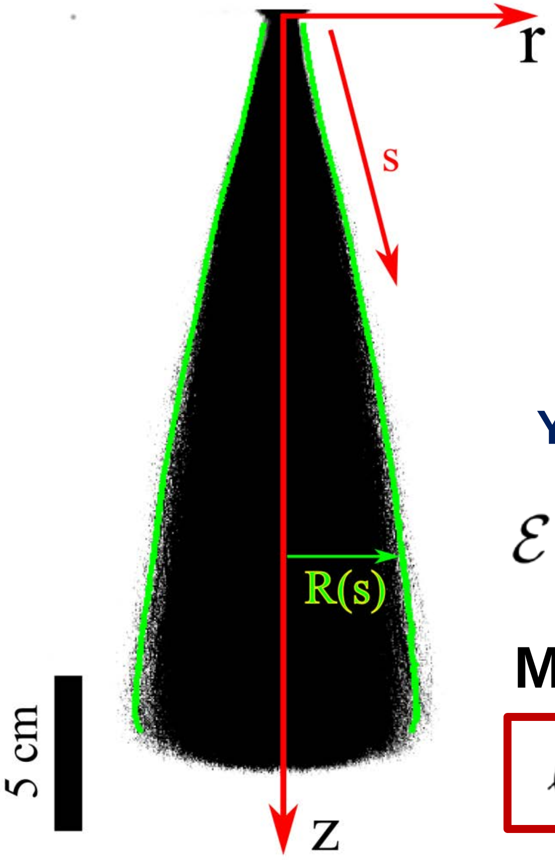
curvature:

$\kappa = |(\mathbf{t} \cdot \nabla)\mathbf{t}|$

pressure:

$P(\rho) = \rho^2 d\langle u \rangle / d\rho$

Application to an Axisymmetric Ponytail



Let n(r,z) be # of fibres within radius r at depth z.

$$2\pi r\rho\sin\theta = -\partial n/\partial z \quad \& \quad 2\pi r\rho\cos\theta = \partial n/\partial r$$

Ansatz of a self-similar density profile:

$$\boxed{n(r,z) = N[r/R(z)]^2} \rightarrow \theta \simeq \frac{r}{R(z)} R_z$$

Yields *an equivalent single-fibre* energy for envelope:

$$\mathcal{E} = N\int_0^L ds\left[\frac{1}{2}\tilde{A}R_{ss}^2 + \frac{1}{2}\tilde{\lambda}g(L-s)R_s^2 + \langle u\rangle\right]$$

Minimization → *The Ponytail Shape Equation*

$$\boxed{\ell^3 R_{ssss} - (L-s)\,R_{ss} + R_s - \Pi(R) = 0}$$

elasticity tension weight pressure

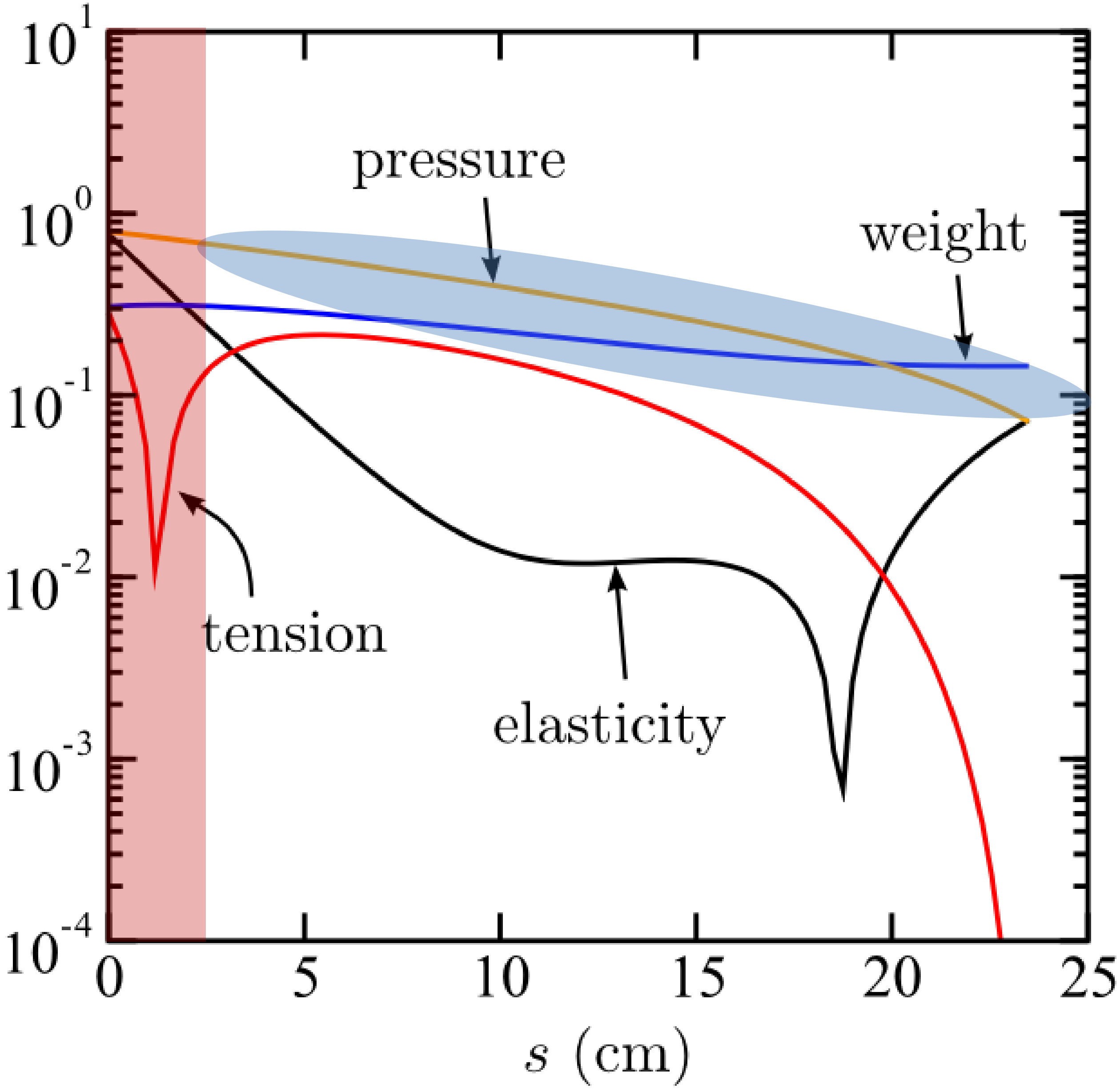
A well-studied problem
(L&L, Audoly & Pomeau)

Van Wyk (1946) – wool
Beckrich *et al*. (2003) - 2D

Average over 5
72º rotations

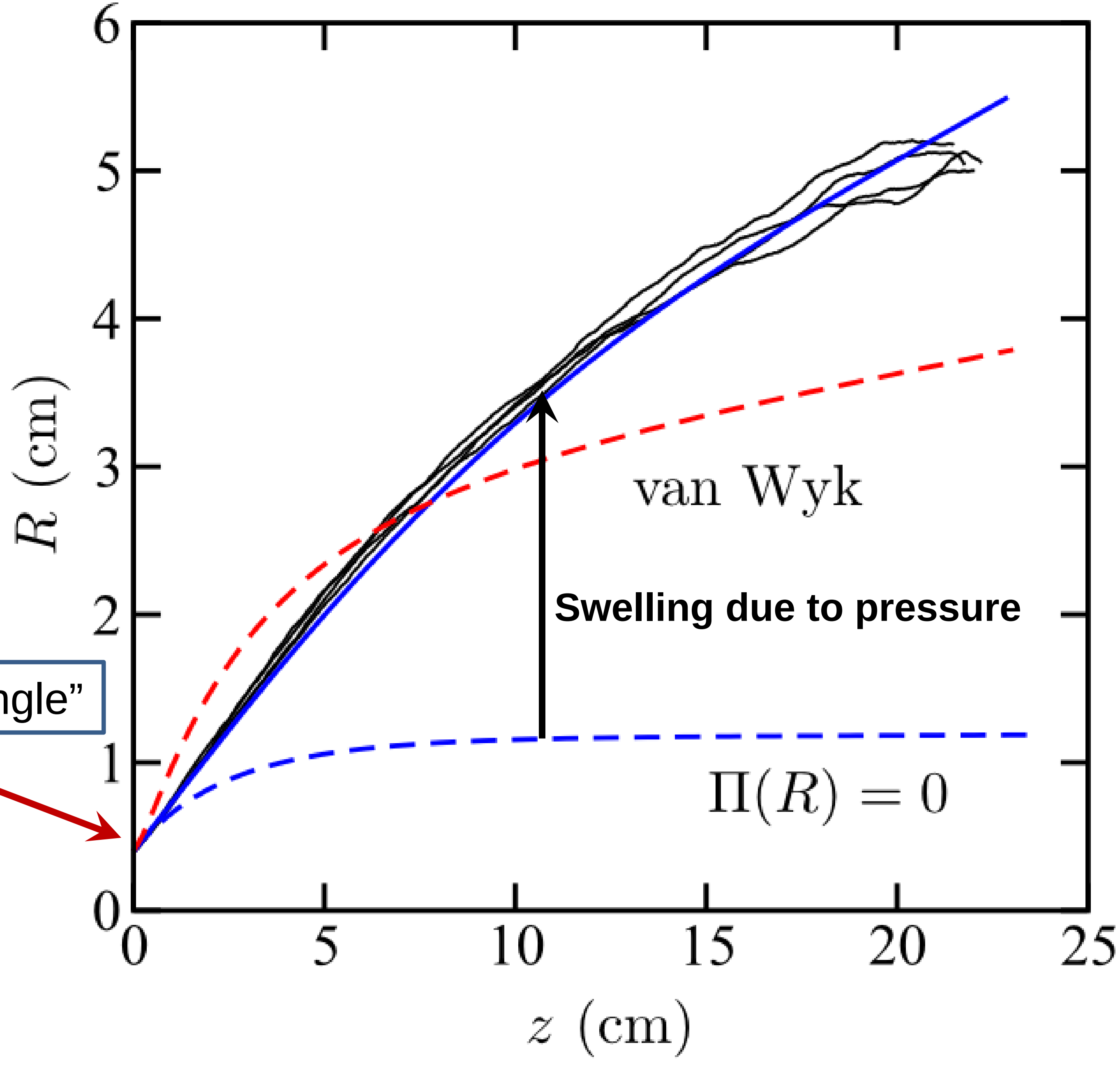
PRL **108**, 078101 (2012)

Balance of Forces Along Length of a Ponytail



PRL **108**, 078101 (2012)

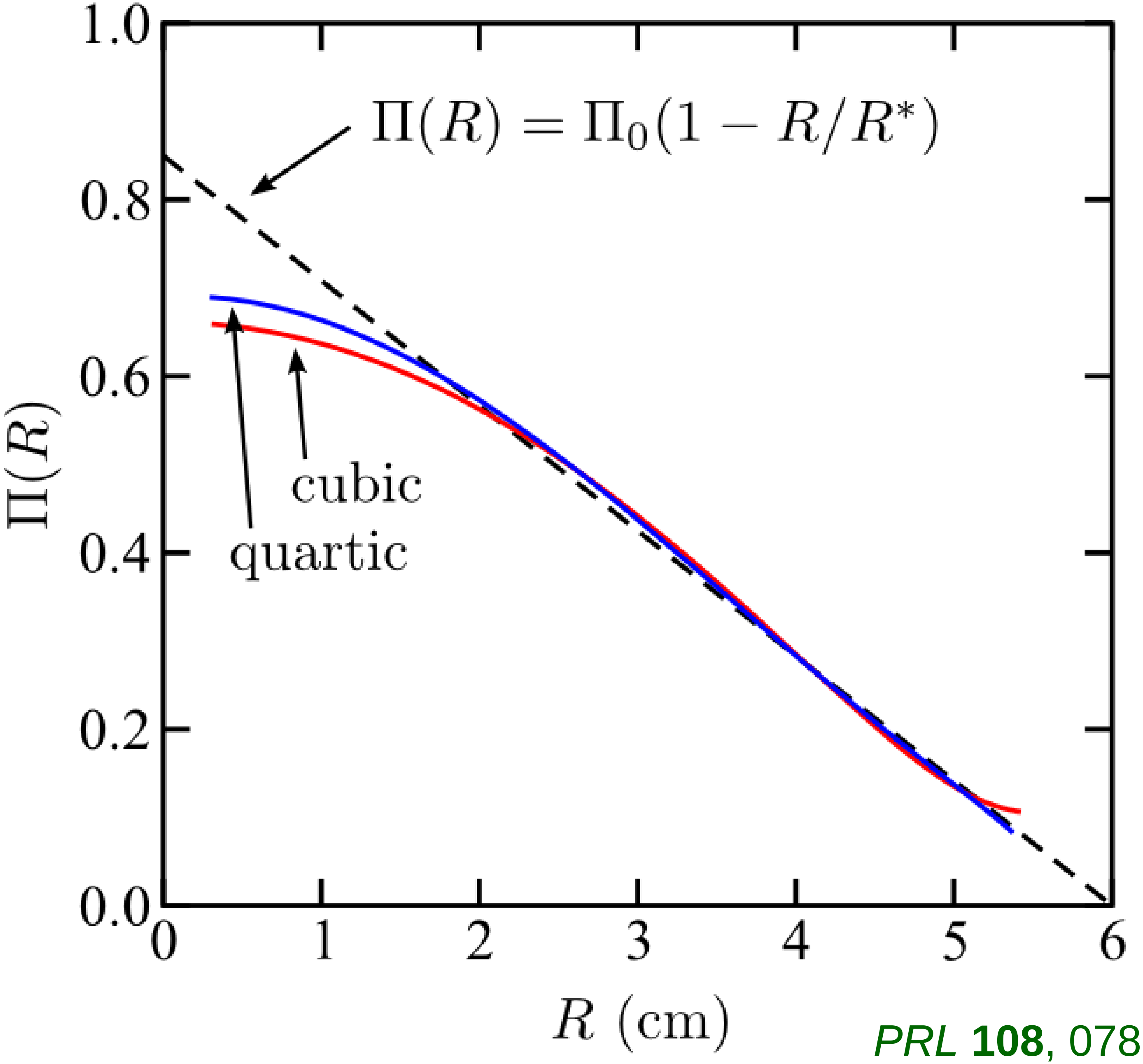
Testing Equations of State



Swelling due to pressure

Fixed “launch angle”

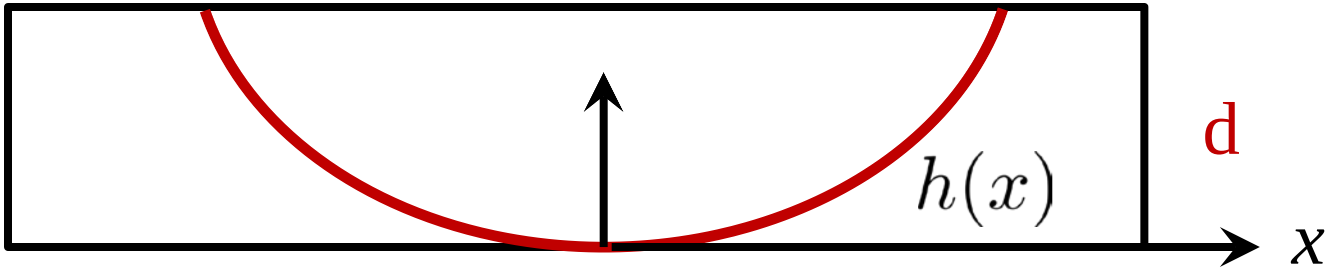
Empirical Equation of State of Hair



PRL **108**, 078101 (2012)

Interpreting the Equation of State

Consistent with the essential features of “tube models”



Elastic energy density with spontaneous curvature:

$$\boxed{e = \tfrac{1}{2} A \left(h_{xx} - \kappa_0\right)^2}$$

parabola $h = \frac{4d}{L^2} x^2$

$$\boxed{e = \tfrac{1}{2} A \kappa_0^2 \left(1 - d/d^*\right)^2}$$

$$d^* = \frac{\kappa_0 L^2}{8}$$

Similar result holds for a helical filament confined to a cylinder:

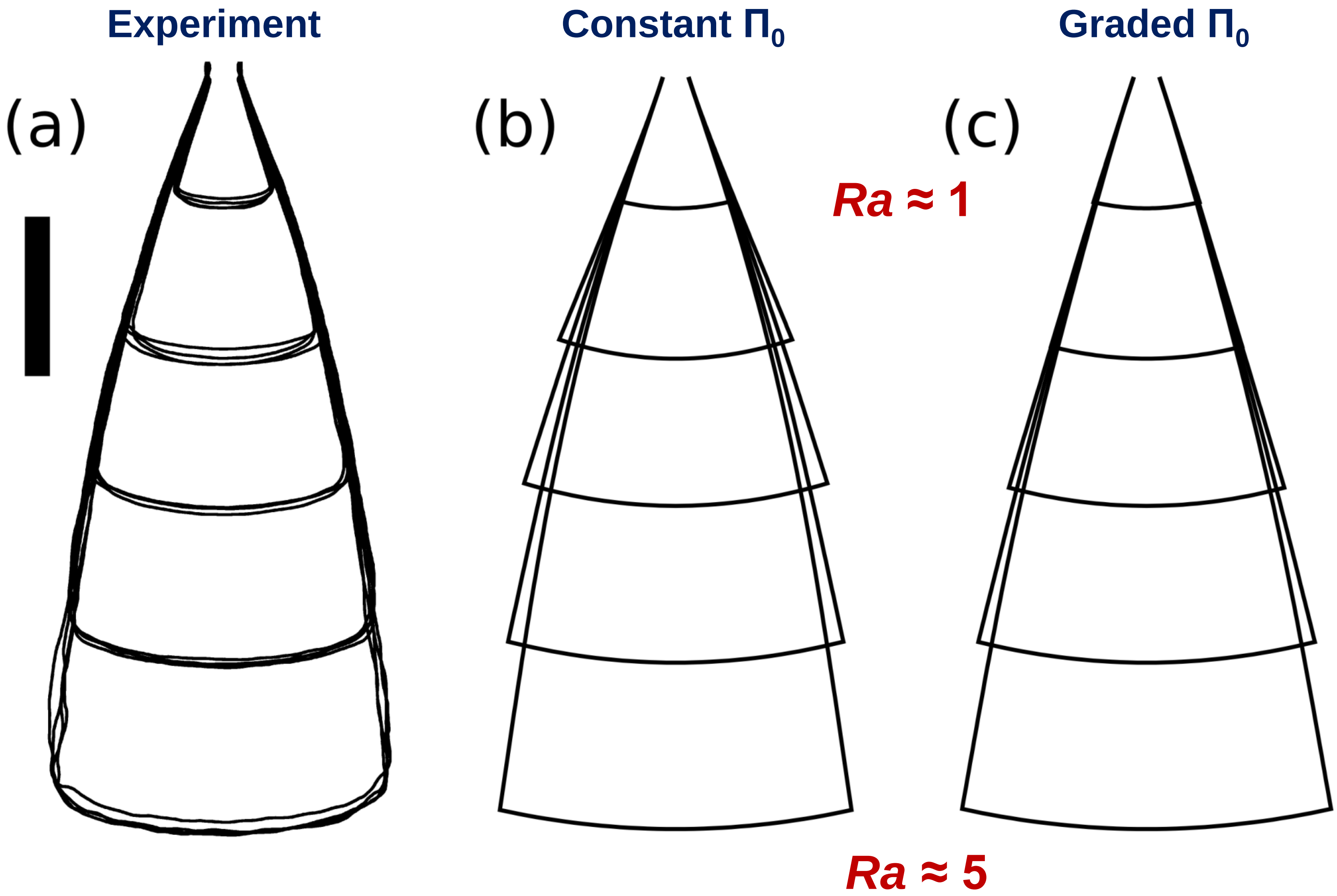
$$\langle u \rangle \approx \tfrac{1}{2} A \langle \kappa_0^2 \rangle \left(1 - a/a_0\right)^2$$

Integrated EOS: $\langle u \rangle = (A/2\ell^3) \int_R^\infty \Pi(R)\, dR$

$$a/a_0 \approx 1 - \alpha + \alpha R/R^* \text{ where } \alpha = \sqrt{\Pi_0 R^* / 2\ell^3 \langle \kappa_0^2 \rangle} \approx 0.4$$

Hence, effective tube is some fraction of the ponytail radius

Experimentum crucis: Trimming Ponytails



Experiment

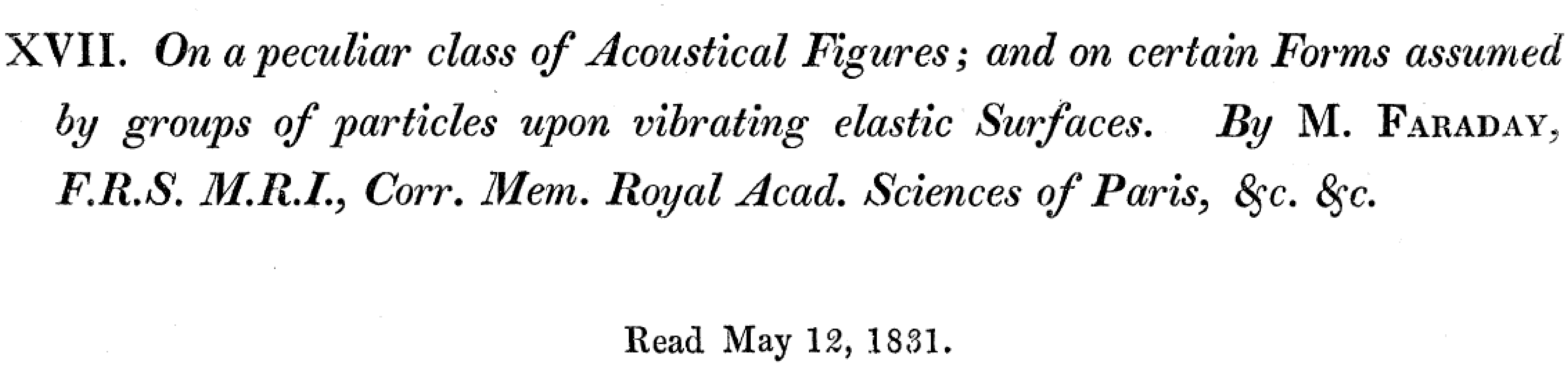
Constant Π_0

Graded Π_0

Ra ≈ 1

Ra ≈ 5

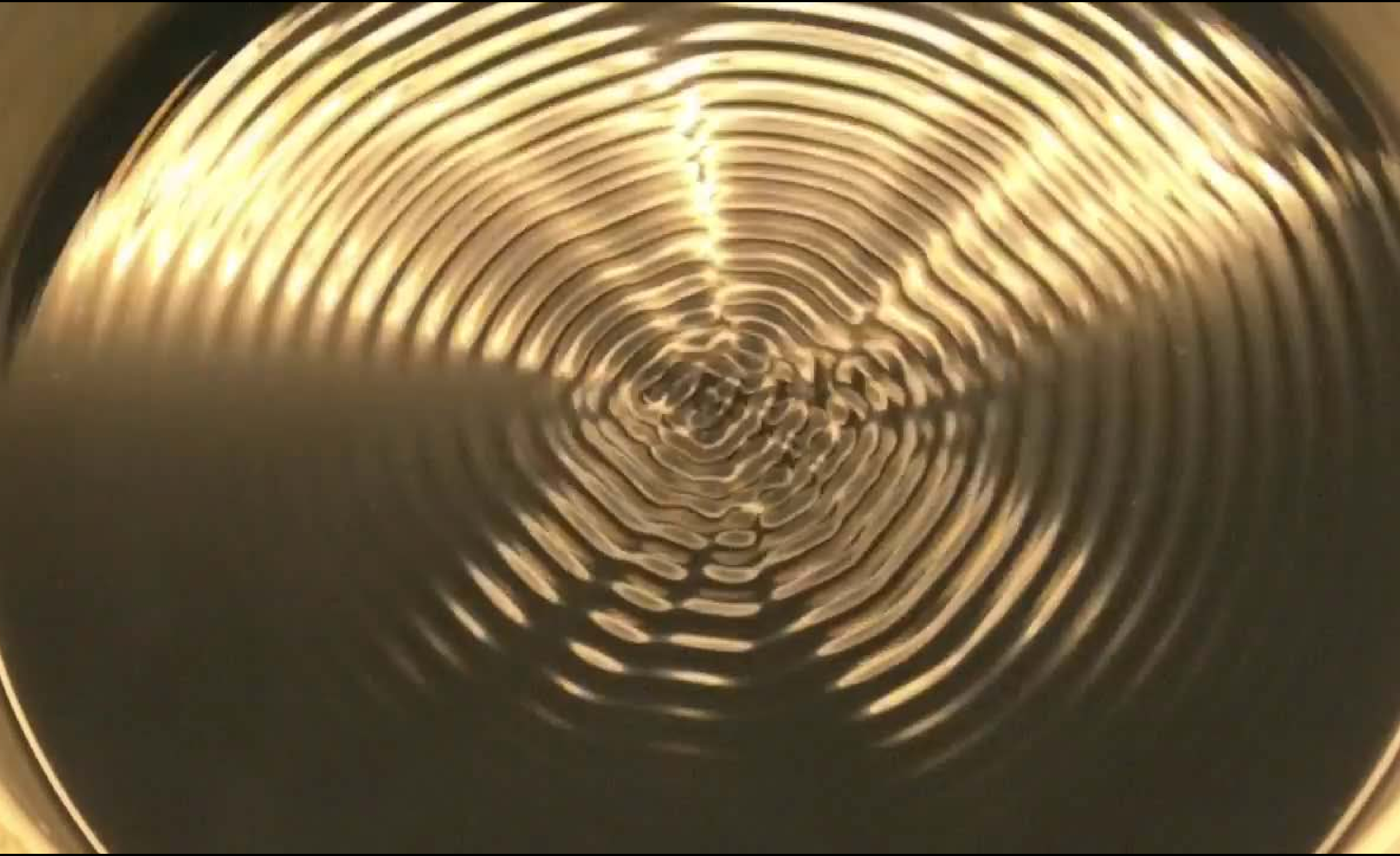
The Faraday Instability



XVII. *On a peculiar class of Acoustical Figures; and on certain Forms assumed by groups of particles upon vibrating elastic Surfaces.* *By* M. FARADAY, *F.R.S. M.R.I., Corr. Mem. Royal Acad. Sciences of Paris, &c. &c.*

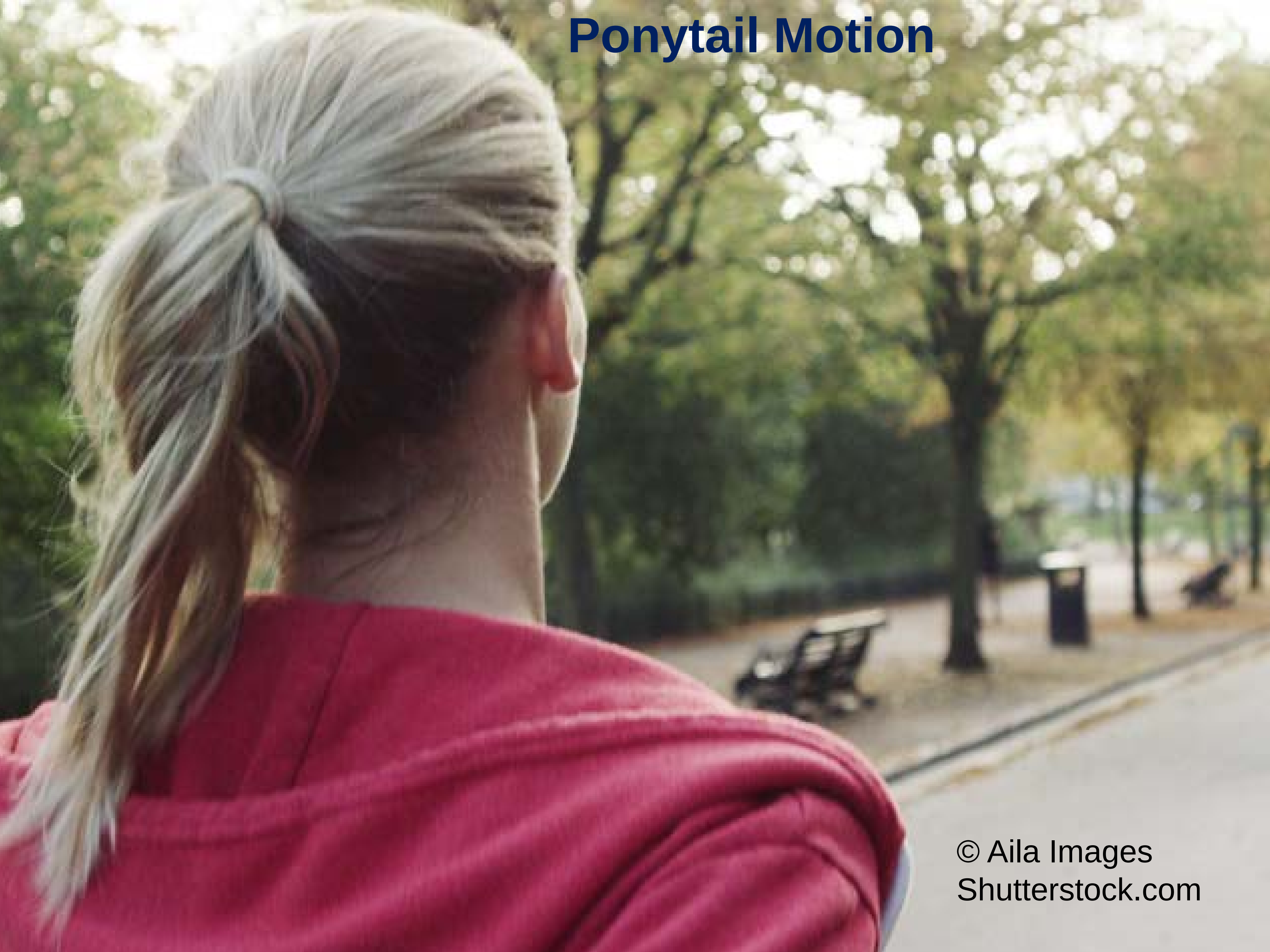
Read May 12, 1831.

Faraday Waves (variable forcing)



Tom Czarnota

Ponytail Motion



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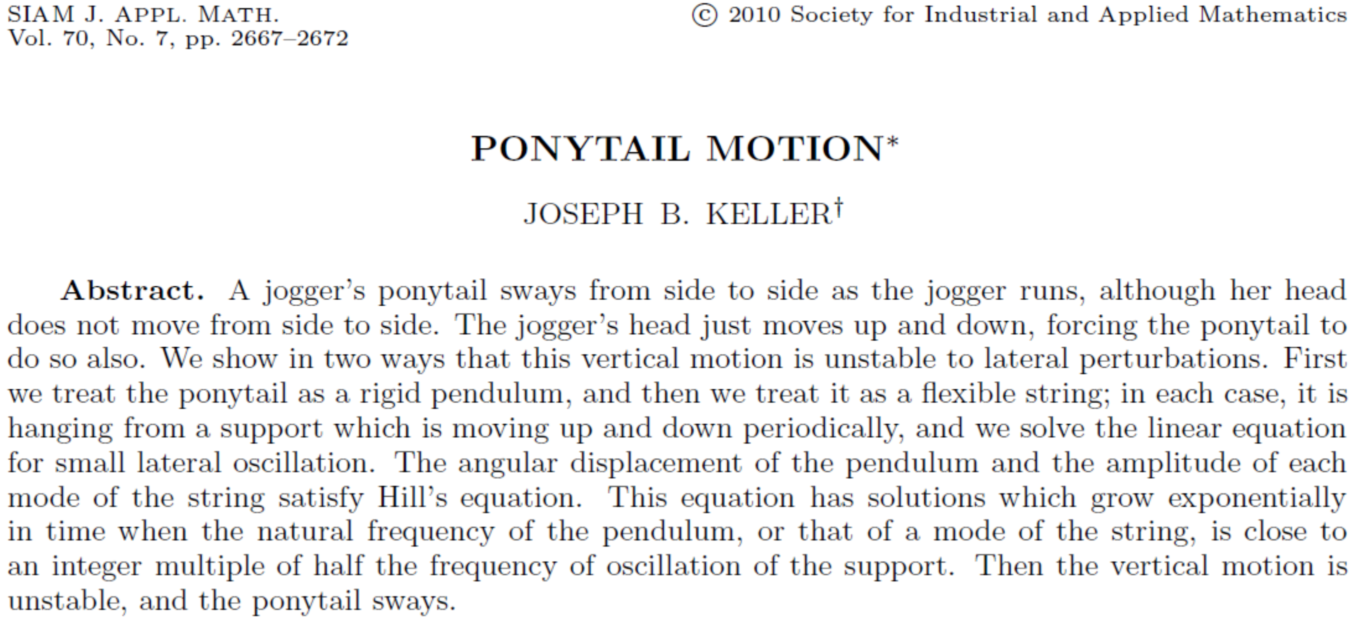
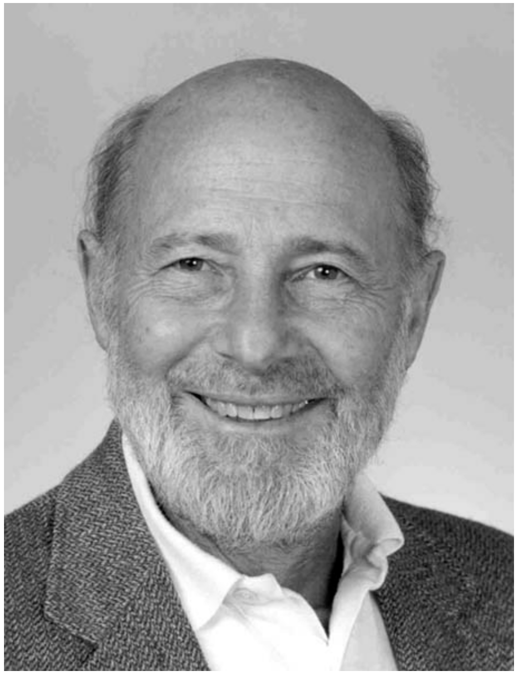


SIAM J. APPL. MATH.
Vol. 70, No. 7, pp. 2667–2672

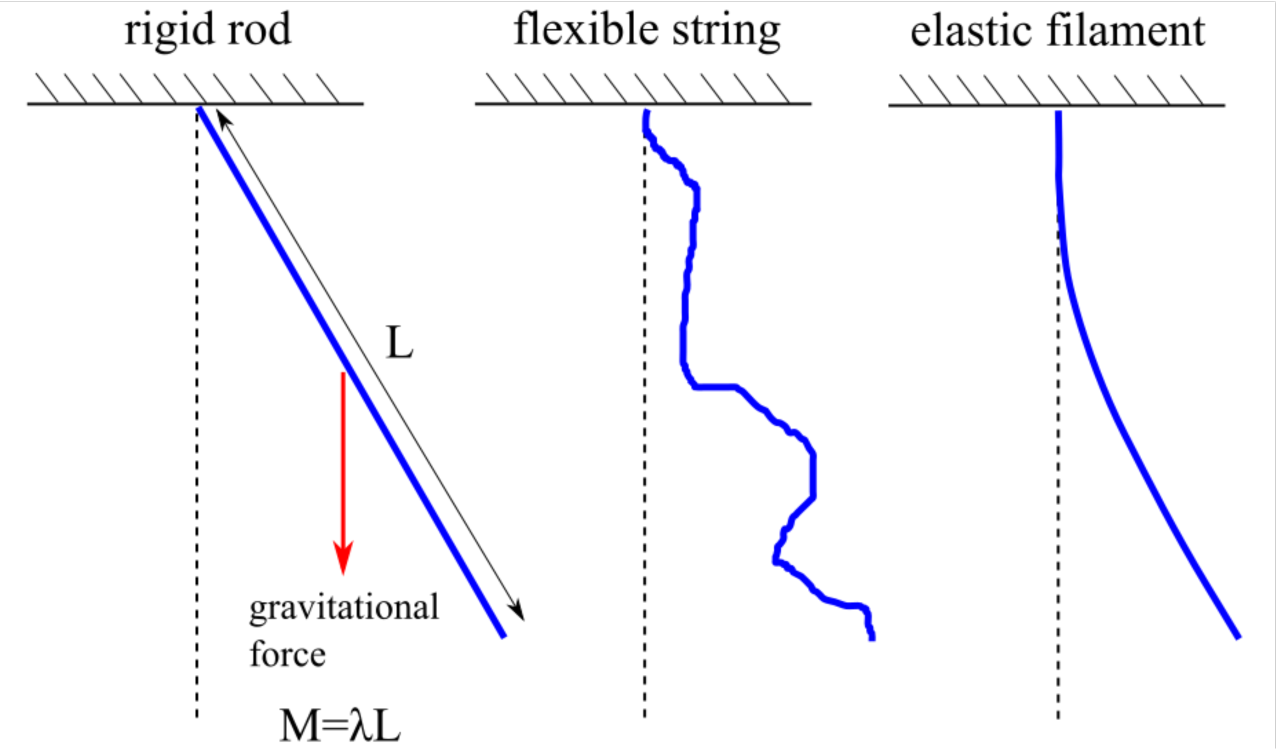
PONYTAIL MOTION*

JOSEPH B. KELLER†

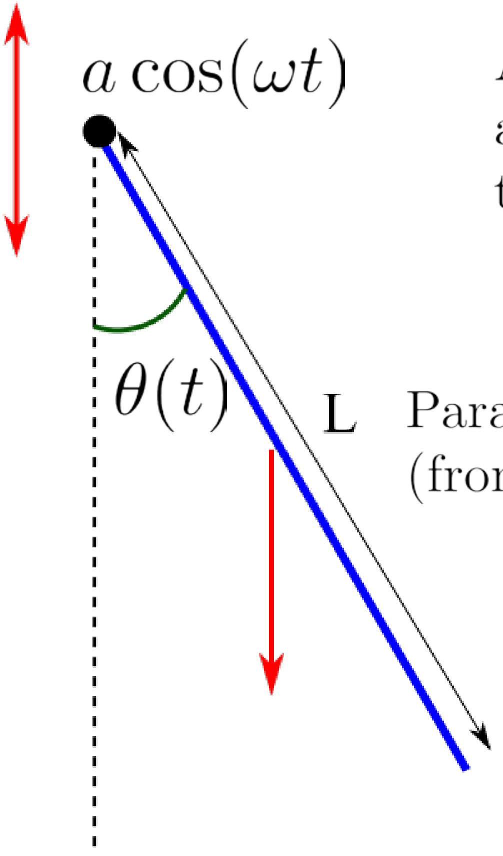
Abstract. A jogger's ponytail sways from side to side as the jogger runs, although her head does not move from side to side. The jogger's head just moves up and down, forcing the ponytail to do so also. We show in two ways that this vertical motion is unstable to lateral perturbations. First we treat the ponytail as a rigid pendulum, and then we treat it as a flexible string; in each case, it is hanging from a support which is moving up and down periodically, and we solve the linear equation for small lateral oscillation. The angular displacement of the pendulum and the amplitude of each mode of the string satisfy Hill's equation. This equation has solutions which grow exponentially in time when the natural frequency of the pendulum, or that of a mode of the string, is close to an integer multiple of half the frequency of oscillation of the support. Then the vertical motion is unstable, and the ponytail sways.



(1923-2016)



The Maths of a “Parametric Excitation”

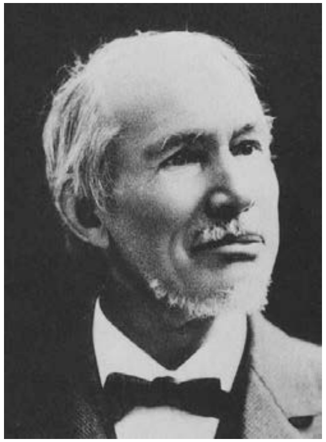


A parameter of the problem, the gravitational acceleration ($g = 980$ cm^2/s), becomes time-dependent:

$$\boxed{g \to g + a\omega^2 \cos(\omega t)}$$

Parametrically forced pendulum equation = Hill’s equation (from studies of the moon’s orbit)

George William Hill



1838-1914

large-amplitude motion when $\frac{\omega_0}{\omega} = \frac{k}{2}$ for $k = 1, 2, 3, \ldots$

Natural frequency:

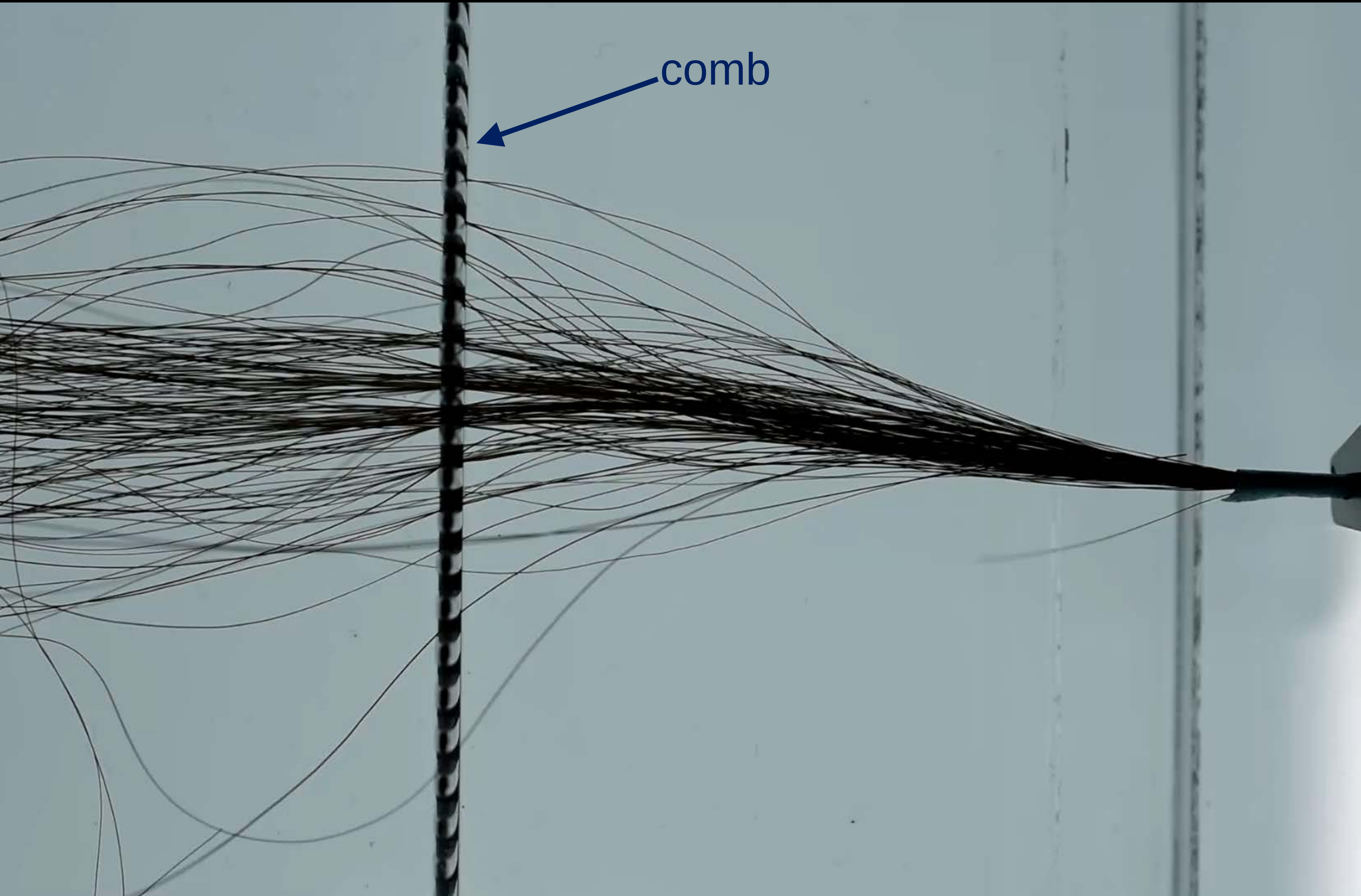
$$\boxed{\omega_0 = \sqrt{\frac{3g}{2L}}}$$

Putting this all together, prediction is that lateral motion will occur for jogging about 140 steps/minute. Spot on.

A Controlled Experiment

A Controlled Experiment

The Next Frontier: Tangling



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